Analytical approach to fractional partial differential equations in fluid mechanics by means of the homotopy perturbation method

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Abstract

Purpose – This paper aims to present a general framework of the homotopy perturbation method (HPM) for analytic treatment of fractional partial differential equations in fluid mechanics. The fractional derivatives are described in the Caputo sense.

Design/methodology/approach – Numerical illustrations that include the fractional wave equation, fractional Burgers equation, fractional KdV equation and fractional Klein-Gordon equation are investigated to show the pertinent features of the technique.

Findings – HPM is a powerful and efficient technique in finding exact and approximate solutions for fractional partial differential equations in fluid mechanics. The implementation of the noise terms, if they exist, is a powerful tool to accelerate the convergence of the solution. The results so obtained reinforce the conclusions made by many researchers that the efficiency of the HPM and related phenomena gives it much wider applicability.

Originality/value – The essential idea of this method is to introduce a homotopy parameter, say p, which takes values from 0 to 1. When $p = 0$, the system of equations usually reduces to a sufficiently simplied form, which normally admits a rather simple solution. As p is gradually increased to 1, the system goes through a sequence of deformations, the solution for each of which is close to that at the previous stage of deformation.

Keywords Deformation, Differential equations, Fluid mechanics, Wave properties

Paper type Research paper

1. Introduction

Fractional calculus has a long history. The idea appeared in a letter by Leibniz to L'Hospital in 1695. However, for three centuries, the theory of fractional calculus was restricted to the field of pure mathematics. In recent years, it has been found that derivatives of non-integer order are very effective for the description of many physical phenomena such as rheology, damping laws, diffusion process. These findings invoked the growing interest of studies of the fractal calculus in various fields such as physics, chemistry and engineering. Recent advances of fractional differential equations are stimulated by new examples of applications which arise in fluid mechanics, viscoelasticity, mathematical biology, electrochemistry and physics. For example, the non-linear oscillation of earthquake can be modeled with fractional derivatives (He, 1998a), and the fluid-dynamic traffic model with fractional derivatives (He, 1999a) can eliminate the deficiency arising from the assumption of continuum traffic flow. Based on experimental data fractional partial differential equations for seepage flow in porous media are suggested in He (1998b), and differential equations with fractional order have recently proved to be valuable tools to the modeling of many physical phenomena (Podlubny, 1999). Different fractional partial differential equations have been studied and solved, the space-time fractional diffusion-wave equation (Al-Khaled and Momani, 2005; Mainardi *et al.*, 2001; Hanyga, 2002), the fractional advection-dispersion equation (Huang and Liu, 2005a, b), the fractional telegraph equation (Momani, 2005a), the

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fractional KdV equation (Momani, 2005b) and the linear inhomogeneous fractional partial differential equations (Debnath and Bhatta, 2004).

Fractional differential equations have been caught much attention recently due to exact description of non-linear phenomena, especially in fluid mechanics, e.g. nano-hydrodynamics, where continuum assumption does not well, and fractional model can be considered to be a best candidate. No analytical method was available before 1998 for such equations even for linear fractional differential equations. In 1998, the variational iteration method was first proposed to solve fractional differential equations with greatest success, see He (1998a). Following the above idea, Draganescu, Momani, Odibat and Das applied the variational iteration method to more complex fractional differential equations, showing effectiveness and accuracy of the used method (Draganescu, 2006; Odibat and Momani, 2006; Das, 2008). In 2002 the Adomian method was suggested to solve fractional differential equations, but many researchers found it is very difficult to calculate the Adomian polynomial (Yusufoğlu, 2007a). Ghorbani and Saberi-Nadjafi (2007) and Ghorbani (2007) suggested a very simple method for calculation the Adomian polynomial using the homotopy perturbation method (HPM), and He polynomial should be used instead of Adomian polynomial. In 2007, Momani and Odibat (2007) applied the HPM to fractional differential equations and revealed the HPM is an alternative analytical method for fractional differential equations. A complete review is available on a review article (He, 2008a).

The objective of this paper is to extend the application of the HPM to obtain analytic solutions to some fractional partial differential equations in fluid mechanics. These equations include wave equations, Burgers equation, KdV equation and Klein-Gordon equation. The HPM is a computational method that yields analytical solutions and has certain advantages over standard numerical methods. It is free from rounding off errors as it does not involve discretization, and does not require large computer obtained memory or power. The method introduces the solution in the form of a convergent fractional series with elegantly computable terms. The corresponding solutions of the integer order equations are found to follow as special cases of those of fractional order equations. Throughout this paper, fractional linear partial differential equations are obtained from the corresponding integer-order equations by replacing the first-order or the second-order time derivative by a fractional in the Caputo sense (Caputo, 1967) of order α with $0 < \alpha \leq 1$ or $1 < \alpha \leq 2$.

The HPM was first proposed by the Chinese mathematician Ji-Huan He (He, 1999b, 2000, 2003, 2004, 2005a, b, 2006a). The essential idea of this method is to introduce a homotopy parameter, say p, which takes values from 0 to 1. When $p = 0$, the system of equations usually reduces to a sufficiently simplied form, which normally admits a rather simple solution. As p is gradually increased to 1, the system goes through a sequence of deformations, the solution for each of which is close to that at the previous stage of deformation. Eventually at $p = 1$, the system takes the original form of the equation and the final stage of deformation gives the desired solution. One of the most remarkable features of the HPM is that usually just few perturbation terms are sufficient for obtaining a reasonably accurate solution. Considerable research works have been conducted recently in applying this method to a class of linear and non-linear equations (Öziş and Yıldırım, 2007a, b, c, d; Yıldırım and Öziş, 2007; Shakeri and Dehghan, 2007, 2008; Dehghan and Shakeri, 2007, 2008a, b, c, d; Saadatmandi *et al.*, 2008; Yusufoğlu, 2007b, c; Chowdhury and Hashim, 2009a, b; Jafari and Momani, 2007; Wang, 2007, 2008; Abdulaziz *et al.*, 2009; Momani and Yıldırım, 2009). The interested reader can see He (2006b, c, 2008b) for last development of HPM. This HPM will become a much more interesting method to solving non-linear differential equations in science and engineering. We extend the method to solve fractional partial differential equations in fluid mechanics.

2. Fractional calculus

We give some basic definitions and properties of the fractional calculus theory which are used further in this paper.

Definition 2.1

A real function $f(x), x > 0$, is said to be in the space $C_\mu, \mu \in R$ if there exists a real number $p(> \mu)$, such that $f(x) = x^p f_1(x)$, where $f_1(x) \in C[0, \infty)$, and it is said to be in the space C_μ^m if $f^{(m)} \in C_\mu, m \in N$.

Definition 2.2

The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, of a function $f \in C_\mu, \mu \geq -1$, is defined as:

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt, \quad \alpha > 0, \quad x > 0,$$
$$J^0 f(x) = f(x).$$

Properties of the operator J^α can be found in Podlubny (1999), Miller and Ross (1993), Samko *et al.* (2007) and Oldham and Spanier (1974), we mention only the following. For $f \in C_\mu, \mu \geq -1, \alpha, \beta \geq 0$ and $\gamma > -1$:

$$J^\alpha J^\beta = J^{\alpha+\beta} f(x), \tag{1}$$

$$J^\alpha J^\beta = J^\beta J^\alpha f(x), \tag{2}$$

$$J^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}. \tag{3}$$

The Riemann-Liouville derivative has certain disadvantages when trying to model real world phenomena with fractional differential equations. Therefore, we shall introduce a modified fractional differential operator D^α proposed by Caputo in his work on the theory of viscoelasticity (Luchko and Gorneflo, 1998).

Definition 2.3

The fractional derivative $f(x)$ in the Caputo sense is defined as:

$$D^\alpha f(x) = J^{m-\alpha} D^m f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f^{(m)}(t)dt, \tag{4}$$

for $m - 1 < \alpha \leq m, m \in N, x > 0, f \in C_{-1}^m$.

Also, we need here two of its basic properties.

Lemma 2.3.1 If $m - 1 < \alpha \leq m, m \in N$ and $f \in C_\mu^m, \mu \geq -1$, then $D^\alpha J^\alpha f(x) = f(x)$, and,

$$J^\alpha D^\alpha f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{x^k}{k!}, \quad x > 0.$$

The Caputo fractional derivatives are considered here because they allow traditional initial and boundary conditions to be included in the formulation of the problem. In this paper, we consider the fractional partial differential equations which arise in fluid mechanics, and the fractional derivatives are taken in Caputo sense as follows.

Definition 2.4

For m to be the smallest integer that exceeds α, the Caputo time-fractional derivative operator of order $\alpha > 0$ is defined as:

$$D_t^\alpha u(x,t) = \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)}\displaystyle\int_0^t (t-\tau)^{m-\alpha-1}\frac{\partial^m u(x,\tau)}{\partial t^m}d\tau, & \text{for } m-1 < \alpha < m \\ \dfrac{\partial^m u(x,t)}{\partial t^m}, & \text{for } \alpha = m \in N \end{cases}. \quad (5)$$

For more information on the mathematical properties of fractional derivatives and integrals, one can consult the mentioned references.

3. Basic ideas of HPM

To illustrate the basic idea of He's HPM, consider the following general non-linear differential equation:

$$A(u) - f(\boldsymbol{r}) = 0, \quad \boldsymbol{r} \in \Omega \quad (6)$$

with boundary conditions,

$$B(u, \partial u/\partial n) = 0, \quad \boldsymbol{r} \in \Gamma \quad (7)$$

where A is a general differential operator, B is a boundary operator, $f(\boldsymbol{r})$ is a known analytic function and Γ is the boundary of the domain Ω.

The operator A can, generally speaking, be divided into two parts L and N, where L is linear, and N is non-linear, therefore Equation (6) can be written as:

$$L(u) + N(u) - f(\boldsymbol{r}) = 0. \quad (8)$$

By using homotopy perturbation technique, one can construct a homotopy $v(r,p) : \Omega \times [0,1] \to R$ which satisfies,

$$H(v,p) = (1-p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0, \quad p \in [0,1], \quad (9a)$$

or

$$H(v,p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0 \quad (9b)$$

where $p \in [0,1]$ is an embedding parameter, and u_0 is the initial approximation of Equation (6) which satisfies the boundary conditions. Clearly, we have,

$$H(v,0) = L(v) - L(u_0) = 0 \quad (10)$$

$$H(v,1) = A(v) - f(r) = 0 \quad (11)$$

the changing process of p from zero to unity is just that of $v(r, p)$ changing from $u_0(r)$ to $u(r)$. This is called deformation, and also, $L(v) - L(u_0)$ and $A(v) - f(r)$ are called homotopic in topology. If, the embedding parameter p; $(0 \leq p \leq 1)$ is considered as a "small parameter", applying the classical perturbation technique, we can naturally assume that the solution of Equations (10) and (11) can be given as a power series in p, i.e.,

$$v = v_0 + pv_1 + p^2 v_2 + \cdots \tag{12}$$

and setting $p = 1$ results in the approximate solution of Equation (9) as:

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \cdots \tag{13}$$

The convergence of series Equation (13) has been proved by He in his paper (He, 2004). It is worth to note that the major advantage of He's HPM is that the perturbation equation can be freely constructed in many ways (therefore is problem dependent) by homotopy in topology and the initial approximation can also be freely selected.

4. Applications

Example 1

Consider the following one-dimensional linear inhomogeneous fractional wave equation (Odibat and Momani, 2006):

$$\frac{\partial^\alpha u}{\partial t^\alpha} + \frac{\partial u}{\partial x} = \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} \sin(x) + t\cos(x), \quad t > 0, x \in R, 0 < \alpha \leq 1, \tag{14}$$

subject to the initial condition:

$$u(x, 0) = 0. \tag{15}$$

To solve Equations (14) and (15) by HPM, we construct the following homotopy:

$$\left(\frac{\partial^\alpha u}{\partial t^\alpha} - \frac{\partial^\alpha u_0}{\partial t^\alpha} \right) = p\left(-\frac{\partial u}{\partial x} + \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} \sin(x) + t\cos(x) - \frac{\partial^\alpha u_0}{\partial t^\alpha} \right), \tag{16}$$

Assume the solution of Equation (16) to be in the form:

$$u = u_0 + pu_1 + p^2 u_2 + p^3 u_3 + \cdots \tag{17}$$

Substituting Equation (17) into Equation (16) and collecting terms of the same power of p give

$$p^0: \quad \frac{\partial^\alpha u_0}{\partial t^\alpha} - \frac{\partial^\alpha u_0}{\partial t^\alpha} = 0, \tag{18}$$

$$p^1: \quad \frac{\partial^\alpha u_1}{\partial t^\alpha} = -\frac{\partial u_0}{\partial x} + \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} \sin(x) + t\cos(x) - \frac{\partial^\alpha u_0}{\partial t^\alpha}, \tag{19}$$

$$p^2: \quad \frac{\partial^\alpha u_2}{\partial t^\alpha} = -\frac{\partial u_1}{\partial x}, \tag{20}$$

$$p^3: \quad \frac{\partial^\alpha u_3}{\partial t^\alpha} = -\frac{\partial u_2}{\partial x}, \tag{21}$$

The given initial value admits the use of:

$$u_0(x,t) = 0. \tag{22}$$

The solution reads:

$$u_1(x,t) = t\sin(x) + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\cos(x), \tag{23}$$

$$u_2(x,t) = -\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\cos(x) + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\sin(x), \tag{24}$$

$$u_3(x,t) = -\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\sin(x) - \frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)}\cos(x), \tag{25}$$

and so on, in this manner the rest of components of the homotopy perturbation series can be obtained.

The solution of Equations (14) and (15) can be obtained by setting $p = 1$ in Equation (17):

$$u = u_0 + u_1 + u_2 + u_3 + \cdots \tag{26}$$

Thus, we have:

$$\begin{aligned} u(x,t) = t\sin(x) &+ \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\cos(x) - \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\cos(x) + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\sin(x) \\ &- \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\sin(x) - \frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)}\cos(x) + \cdots. \end{aligned} \tag{27}$$

It is easily observed that the self-canceling "noise" terms appear between various components. Canceling the noise terms and keeping the non-noise terms in Equation (27) yield the exact solution of Equations (14) and (15) given by:

$$u(x,t) = t\sin(x), \tag{28}$$

which is easily verified. It is worth noting that other noise terms between other components of Equation (27) will be canceled, as the sixth terms, and the sum of these "noise" terms will vanish in the limit.

Example 2

In this example, we consider the one-dimensional linear inhomogeneous fractional Burgers equation given by (Odibat and Momani, 2006):

$$\frac{\partial^\alpha u}{\partial t^\alpha} + \frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2} = \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} + 2x - 2, \quad t > 0, x \in R, 0 < \alpha \leq 1, \tag{29}$$

subject to the initial condition,

$$u(x,0) = x^2. \tag{30}$$

To solve Equations (29) and (30) by HPM, we construct the following homotopy:

$$\left(\frac{\partial^\alpha u}{\partial t^\alpha} - \frac{\partial^\alpha u_0}{\partial t^\alpha}\right) = p\left(-\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} + \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} + 2x - 2 - \frac{\partial^\alpha u_0}{\partial t^\alpha}\right) \tag{31}$$

Substituting Equation (17) into Equation (31) and collecting terms of the same power of p give:

$$p^0: \quad \frac{\partial^\alpha u_0}{\partial t^\alpha} - \frac{\partial^\alpha u_0}{\partial t^\alpha} = 0, \tag{32}$$

$$p^1: \quad \frac{\partial^\alpha u_1}{\partial t^\alpha} = -\frac{\partial u_0}{\partial x} + \frac{\partial^2 u_0}{\partial x^2} + \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} + 2x - 2 - \frac{\partial^\alpha u_0}{\partial t^\alpha}, \tag{33}$$

$$p^2: \quad \frac{\partial^\alpha u_2}{\partial t^\alpha} = -\frac{\partial u_1}{\partial x} + \frac{\partial^2 u_1}{\partial x^2}, \tag{34}$$

$$p^3: \quad \frac{\partial^\alpha u_3}{\partial t^\alpha} = -\frac{\partial u_2}{\partial x} + \frac{\partial^2 u_2}{\partial x^2}. \tag{35}$$

The given initial value admits the use of:

$$u_0(x,t) = x^2. \tag{36}$$

The solution reads:

$$u_1(x,t) = t^2, \tag{37}$$

$$u_2(x,t) = 0, \tag{38}$$

$$u_3(x,t) = 0, \tag{39}$$

and so on, in this manner the rest of components of the homotopy perturbation series can be obtained.

Thus, we have,

$$u(x,t) = x^2 + t^2, \tag{40}$$

which is the exact solution of the problem.

Example 3

We consider the one-dimensional linear inhomogeneous fractional Klein-Gordon equation (Odibat and Momani, 2006):

$$\frac{\partial^\alpha u}{\partial t^\alpha} - \frac{\partial^2 u}{\partial x^2} + u = 6x^3 t + (x^3 - 6x)t^3, \quad t > 0, x \in R, 1 < \alpha \leq 2, \tag{41}$$

subject to the initial conditions,

$$u(x,0) = 0, \quad u_t(x,0) = 0. \tag{42}$$

To solve Equations (41) and (42) by HPM, we construct the following homotopy:

$$\left(\frac{\partial^\alpha u}{\partial t^\alpha} - \frac{\partial^\alpha u_0}{\partial t^\alpha}\right) = p\left(\frac{\partial^2 u}{\partial x^2} - u + 6x^3 t + (x^3 - 6x)t^3 - \frac{\partial^\alpha u_0}{\partial t^\alpha}\right). \tag{43}$$

Substituting Equation (17) into Equation (43) and collecting terms of the same power of p give:

$$p^0: \quad \frac{\partial^\alpha u_0}{\partial t^\alpha} - \frac{\partial^\alpha u_0}{\partial t^\alpha} = 0 \tag{44}$$

$$p^1: \quad \frac{\partial^\alpha u_1}{\partial t^\alpha} = \frac{\partial^2 u_0}{\partial x^2} - u_0 + 6x^3 t + (x^3 - 6x)t^3 - \frac{\partial^\alpha u_0}{\partial t^\alpha}, \tag{45}$$

$$p^2: \quad \frac{\partial^\alpha u_2}{\partial t^\alpha} = \frac{\partial^2 u_1}{\partial x^2} - u_1, \tag{46}$$

$$p^3: \quad \frac{\partial^\alpha u_3}{\partial t^\alpha} = \frac{\partial^2 u_2}{\partial x^2} - u_2. \tag{47}$$

The given initial values admit the use of:

$$u_0(x,t) = 0, \tag{48}$$

The solution reads:

$$\begin{aligned} u_1(x,t) &= 6x^3 \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + (x^3 - 6x)\frac{6t^{\alpha+3}}{\Gamma(\alpha+4)}, \\ u_2(x,t) &= 36x\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} + 36x\frac{t^{2\alpha+3}}{\Gamma(2\alpha+4)} - 6x^3\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} \\ &\quad - (x^3 - 6x)\frac{6t^{2\alpha+3}}{\Gamma(2\alpha+4)}, \end{aligned} \tag{50}$$

and so on, in this manner the rest of components of the homotopy perturbation series can be obtained. Thus, we have:

$$\begin{aligned} u(x,t) = 6x^3 \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + (x^3 - 6x)\frac{6t^{\alpha+3}}{\Gamma(\alpha+4)} + 36x\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} + 36x\frac{t^{2\alpha+3}}{\Gamma(2\alpha+4)} \\ - 6x^3\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} - (x^3 - 6x)\frac{6t^{2\alpha+3}}{\Gamma(2\alpha+4)} + \cdots. \end{aligned} \tag{51}$$

When $\alpha = 2$, we obtain the solution for the classical Klein-Gordon equation which is given by:

$$\begin{aligned} u(x,t) = x^3t^3 + (x^3 - 6x)\frac{6t^5}{\Gamma(6)} + 36x\frac{t^5}{\Gamma(6)} + 36x\frac{t^7}{\Gamma(8)} - 6x^3\frac{t^5}{\Gamma(6)} \\ - (x^3 - 6x)\frac{6t^7}{\Gamma(8)} + \cdots. \end{aligned} \tag{52}$$

Canceling the noise terms and keeping the non-noise terms in Equation (52) yield the exact solution of Equations (41) and (42), for this special case, given by,

$$u(x,t) = x^3t^3, \tag{53}$$

which is easily verified.

Example 4
We consider the time-fractional KdV equation (Momani, 2005a):

$$\frac{\partial^\alpha u}{\partial t^\alpha} + 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0, \quad t > 0, x \in R, 0 < \alpha \leq 1, \tag{54}$$

subject to the initial condition,

$$u(x,0) = \frac{1}{2}\sec h^2\left(\frac{1}{2}x\right). \tag{55}$$

The exact solution, for the special case $\alpha = 1$, is given by:

$$u(x,t) = \frac{1}{2}\sec h^2\left(\frac{1}{2}(x-t)\right). \tag{56}$$

To solve Equations (54) and (55) by HPM, we construct the following homotopy:

$$\left(\frac{\partial^\alpha u}{\partial t^\alpha} - \frac{\partial^\alpha u_0}{\partial t^\alpha}\right) = p\left(-6u\frac{\partial u}{\partial x} - \frac{\partial^3 u}{\partial x^3} + \frac{\partial^\alpha u_0}{\partial t^\alpha}\right), \tag{57}$$

Substituting Equation (17) into Equation (57) and collecting terms of the same power of p give:

$$p^0: \quad \frac{\partial^\alpha u_0}{\partial t^\alpha} - \frac{\partial^\alpha u_0}{\partial t^\alpha} = 0, \tag{58}$$

$$p^1: \quad \frac{\partial^\alpha u_1}{\partial t^\alpha} = -6u_0\frac{\partial u_0}{\partial x} - \frac{\partial^3 u_0}{\partial x^3} + \frac{\partial^\alpha u_0}{\partial t^\alpha}, \tag{59}$$

$$p^2: \quad \frac{\partial^\alpha u_2}{\partial t^\alpha} = -6u_0\frac{\partial u_1}{\partial x} - 6u_1\frac{\partial u_0}{\partial x} - \frac{\partial^3 u_1}{\partial x^3}, \tag{60}$$

$$p^3: \quad \frac{\partial^\alpha u_3}{\partial t^\alpha} = -6u_0\frac{\partial u_2}{\partial x} - 6u_1\frac{\partial u_1}{\partial x} - 6u_2\frac{\partial u_0}{\partial x} - \frac{\partial^3 u_2}{\partial x^3}. \tag{61}$$

The given initial values admit the use of:

$$u_0(x,t) = \frac{1}{2}\sec h^2\left(\frac{1}{2}x\right). \tag{62}$$

The solution reads:

$$u_1(x,t) = (-6u_0u_0' - u_0''')\frac{t^\alpha}{\Gamma(\alpha+1)},$$

$$u_2(x,t) = (-6u_1u_0' - 6u_0u_1' - u_1''')\frac{t^{2\alpha}}{\Gamma(2\alpha+1)},$$

$$u_3(x,t) = (-6u_2u_0' - 6u_1u_1' - 6u_0u_2' - u_2''')\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}$$

and so on, in this manner the rest of components of the homotopy perturbation series can be obtained. Thus, we have:

$$u(x,t) = f_0(x) + f_1(x)\frac{t^{\alpha}}{\Gamma(\alpha+1)} + f_2(x)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + f_3(x)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots, \quad (63)$$

where,

$$\begin{aligned}
f_0(x) &= \frac{1}{2}\sec h^2\left(\frac{1}{2}x\right),\\
f_1(x) &= -6f_0f_0 - f_0''',\\
f_2(x) &= -6f_1f_0' - 6f_0f_1' - f_1''',\\
f_3(x) &= -6f_2f_0' - 6f_1f_1' - 6f_0f_2' - f_2'''.
\end{aligned}$$

For the special case $\alpha = 1$, we obtain:

$$\begin{aligned}
u(x,t) = {}& \frac{1}{2}\sec h^2\left(\frac{1}{2}x\right) + \frac{1}{2}\sec h^2\left(\frac{1}{2}x\right)\tanh\left(\frac{1}{2}x\right)t\\
&+ \left(\frac{1}{2}\sec h^2\left(\frac{1}{2}x\right) - \frac{3}{4}\sec h^4\left(\frac{1}{2}x\right)\right)\frac{t^2}{2} + \cdots.
\end{aligned} \quad (64)$$

This series has the closed form:

$$u(x,t) = \frac{1}{2}\sec h^2\left(\frac{1}{2}(x-t)\right) \quad (65)$$

which is the exact solution of the problem.

Example 5

In this example, we consider the time-fractional Boussinesq-like equation (Odibat and Momani, 2006):

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} + \frac{\partial^2(u^2)}{\partial x^2} - \frac{\partial^4(u^2)}{\partial x^4} = 0, \quad t > 0, x \in R, 1 < \alpha \leq 2, \quad (66)$$

subject to the initial conditions:

$$u(x,0) = \frac{4}{3}\sinh^2\left(\frac{1}{4}x\right), \quad u_t(x,0) = -\frac{1}{3}\sinh\left(\frac{1}{2}x\right). \quad (67)$$

To solve Equations (66) and (67) by HPM, we construct the following homotopy:

$$\left(\frac{\partial^{\alpha} u}{\partial t^{\alpha}} - \frac{\partial^{\alpha} u_0}{\partial t^{\alpha}}\right) = p\left(-\frac{\partial^2(u^2)}{\partial x^2} + \frac{\partial^4(u^2)}{\partial x^4} + \frac{\partial^{\alpha} u_0}{\partial t^{\alpha}}\right). \quad (68)$$

Substituting Equation (17) into Equation (67) and collecting terms of the same power of p give:

$$p^0: \quad \frac{\partial^\alpha u_0}{\partial t^\alpha} - \frac{\partial^\alpha u_0}{\partial t^\alpha} = 0, \tag{69}$$

$$p^1: \quad \frac{\partial^\alpha u_1}{\partial t^\alpha} = -\frac{\partial^2(u_0^2)}{\partial x^2} + \frac{\partial^4(u_0^2)}{\partial x^4} + \frac{\partial^\alpha u_0}{\partial t^\alpha}, \tag{70}$$

$$p^2: \quad \frac{\partial^\alpha u_2}{\partial t^\alpha} = -\frac{\partial^2(2u_0u_1)}{\partial x^2} + \frac{\partial^4(2u_0u_1)}{\partial x^4}, \tag{71}$$

$$p^3: \quad \frac{\partial^\alpha u_2}{\partial t^\alpha} = -\frac{\partial^2(2u_0u_2 + u_1^2)}{\partial x^2} + \frac{\partial^4(2u_0u_2 + u_1^2)}{\partial x^4}. \tag{72}$$

The given initial values admit the use of:

$$u_0(x,t) = \frac{4}{3}\sinh^2\left(\frac{1}{4}x\right) - \frac{1}{3}\sinh\left(\frac{1}{2}x\right)t. \tag{73}$$

Solving Equation (66) recursively, as a result we obtain the following approximate solution for the time-fractional Boussinesq-like Equation (66):

$$\begin{aligned} u(x,t) = {} & \frac{4}{3}\sinh^2\left(\frac{1}{4}x\right) - \frac{1}{3}\sinh\left(\frac{1}{2}x\right)t + \frac{1}{2.3}\cosh\left(\frac{1}{2}x\right)\frac{t^\alpha}{\Gamma(\alpha+1)} \\ & - \frac{1}{2^2 3}\sinh\left(\frac{1}{2}x\right)\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{1}{2^3.3}\cosh\left(\frac{1}{2}x\right)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \\ & - \frac{1}{2^4 3}\sinh\left(\frac{1}{2}x\right)\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} + \frac{1}{2^5.3}\cosh\left(\frac{1}{2}x\right)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} \\ & - \frac{1}{2^6 3}\sinh\left(\frac{1}{2}x\right)\frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)} + \cdots. \end{aligned} \tag{74}$$

For the special case $\alpha = 2$, we obtain:

$$\begin{aligned} u(x,t) = {} & \frac{2}{3}\left[\cosh\left(\frac{1}{2}x\right)\left(1 + \frac{t^2}{2^2 2!} + \frac{t^4}{2^4 4!} + \ldots\right) - 1\right] \\ & - \frac{2}{3}\sinh\left(\frac{1}{2}x\right)\left[\frac{1}{2}t + \frac{t^3}{2^3 3!} + \frac{t^5}{2^5 5!} + \ldots\right]. \end{aligned} \tag{75}$$

This series has the closed form:

$$u(x,t) = \frac{4}{3}\sinh^2\left(\frac{1}{4}(x-t)\right). \tag{76}$$

5. Conclusions

HPM is a powerful and efficient technique in finding exact and approximate solutions for fractional partial differential equations in fluid mechanics. The implementation of the noise terms, if exist, is a powerful tool to accelerate the convergence of the solution. The results so obtained reinforce the conclusions made by many researchers that the efficiency of the HPM and related phenomena gives it much wider applicability. A disadvantage of this new approach is to need an initial value. This technique cannot be employed if the problem does not include initial and boundary conditions.

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